Bruce Sagan

The Protean Chromatic Polynomial

I am very excited to have the opportunity to share some of the ideas surrounding one of my favorite objects in combinatorics, the chromatic polynomial, during my Invited Address. Let me start by defining each of the terms in my title.

The Merriam-Webster Dictionary defines "protean" as "of or resembling Proteus in having a varied nature or ability to assume different forms." In Greek mythology, Proteus was one of the gods of the sea and thus was associated with its constantly changing nature. In a similar manner, the chromatic polynomial gives one information about many things which, a priori, have nothing to do with its original purpose, as described below.

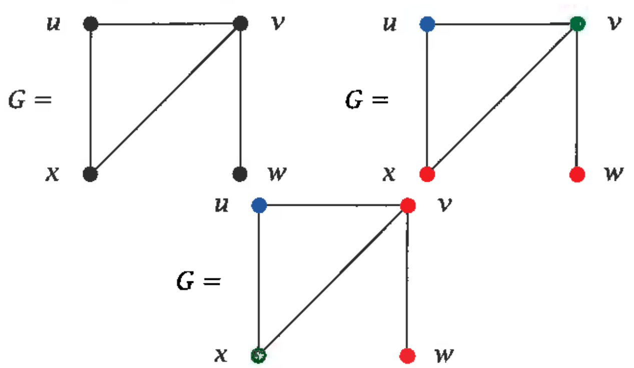

Figure 1. A graph and two colorings

"Chromatic" refers to color, and our general topic is the coloring of the vertices of a graph. A *(combinatorial) graph*, G, consists of a set of vertices V and a set of edges E which connect pairs of vertices. For example, the graph in the upper left in Figure 1 has vertex set $V = \{u, v, w, x\}$ and edge set $E = \{uv, ux, vx, vw\}$. A *coloring* of G is a function $c : V \to S$ where S is called the *color set*. The coloring is *proper* if the endpoints of every edge have different colors. The coloring in the upper right of Figure 1 is proper, while the one on the bottom is not because the edge $e = vw$ has the same color on both endpoints. The *chromatic number*, $\chi(G)$, is the smallest number of colors needed to properly color G. The graph in Figure 1 has $\chi(G) = 3$ since the upper right image exhibits a proper coloring with three colors, and the triangle uvx cannot be colored with fewer colors. Maybe the most famous theorem in graph theory is the Four Color Theorem, which states that if a graph is planar (can be drawn in the plane without edge crossings), then $\chi(G) \leq 4$. This statement was a conjecture for over a hundred years until it was finally proved by Wolfgang Haken and Kenneth Appel in 1976. Their proof caused quite a stir in the mathematical community, because it was the first to use a substantial amount of computing time, and the large number of cases could not all be checked by hand.

The chromatic polynomial was introduced in 1912 by George Birkhoff as a possible tool for proving the then Four Color Conjecture. Although it did not turn out to be useful for the eventual proof, it has more than justified its existence through its many other applications. Let t be a nonnegative integer. The *chromatic polynomial*, $P(G;t)$, is the number of proper colorings $c : V \to \{1, 2 \ldots, t\}$. It is not apparent at first blush why this cardinality should be called a polynomial. However, this will become clearer if we compute $P(G;t)$ for the graph in Figure 1. Suppose we color the vertices in the order u, v, w, x. Then there are t choices for the color of u since it is the first vertex to be colored. After that, there will be $t-1$ choices for the color of v, since it cannot be the same color as u. Similar reasoning shows that there are $t-1$ choices for w. Finally, x is adjacent to both u and v, and these two vertices have different colors, so we can color x in $t-2$ ways. The net result is that

$$P(G;t) = t(t-1)^2(t-2) = t^4 - 4t^3 + 5t^2 - 2t,$$

which is a polynomial in t, the number of colors!

One can show that $P(G;t)$ is always a polynomial in t and give nice characterizations of its degree, coefficients, and other properties. Furthermore, it has connections with many other objects of study, including acyclic orientations of graphs, hyperplane arrangements, and even Chern classes in algebraic geometry. I will explain these during my lecture, as well as present some recent work with Joshua Hallam and Jeremy Martin relating $P(G;t)$ to yet another graphical concept, increasing spanning forests. If you are at the Buffalo AMS Sectional Meeting in September, I hope to see you at my talk.

Image Credits

Figure 1 courtesy of Bruce Sagan.
Photo of Bruce Sagan by Robert Chandler, courtesy of Bruce Sagan.

ABOUT THE AUTHOR

Bruce Sagan does research in enumerative, algebraic, and geometric combinatorics. He is best known for his book *The Symmetric Group*. He has twice received the MSU mathematics department's Frame Teaching Excellence Award. When not doing mathematics, Bruce plays folk music. The photograph shows him playing the Swedish nyckelharpa.

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